Nice Math

Abdullah Al Mahmud

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Contents

Paradoxes

Rope Around Equator

A rope around earth equator needs only 2 m more to rise 31 cm higher from equator

Yep—that paradox-y claim is (approximately) right.

If a rope encircles a perfect circle and you want it to hover uniformly by height h, the extra length needed is

$$\Delta L = 2\pi h,$$

independent of the circle's size.

Plug in $h = 0.31$ m:

$$\Delta L = 2\pi(0.31) \approx 0.62\pi \approx 1.948 \text{ m} \approx 2 \text{ m}.$$

So adding about 2 meters lets the rope sit ~31 cm above the equator all the way around—assuming a perfectly circular Earth and a taut, uniformly lifted rope (no sag).

Detailed

$$C = 2\pi r, \quad C' = 2\pi(r + h) = 2\pi r + 2\pi h.$$

So the *extra rope length* is

$$\Delta C = C' - C = 2\pi h.$$

Now, **the logic** behind it (without formulas):

- Think of the rope not as "tied to Earth" but as encircling a circle of radius r.
- If you want to lift it by the same gap h everywhere, the circle it forms just becomes a slightly bigger circle of radius $r + h$.
- The "extra rope" you need is exactly the circumference of that skinny ring (a circle of radius h), i.e. one tiny circle wrapped around the original one.

- That's why it's $2\pi h$ — and importantly, **it doesn't depend on Earth's radius at all**.

So whether it's Earth, a basketball, or a coffee mug, adding ~2 m to a rope lets you lift it 31 cm off the surface all the way around.

The *counterintuitive part* is that people expect the Earth's huge size to matter, but it cancels out — the difference comes only from the uniform lift height h.

Would you like me to also give you a **visual analogy** (like using a belt around your waist) to cement the intuition?

$0.999\cdots = 1$

Proof 01

Let $X = 0.999\cdots$

- $10X = 10 \times 0.999\cdots = 9.999\cdots$
- $9X = 9.000\cdots$
- $X = 1$

Proof 02

- $\frac{1}{9} = 0.111\cdots$
- $9 \times \frac{1}{9} = 9 \times 0.111\cdots$
- $1 = 0.999\cdots$

Proof 03

- $0.999\cdots = 0.9 + 0.09 + 0.009 + \cdots$
- $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

Here, common ratio $r = \frac{1}{10} < 1$

Thus, summation $S = \frac{a}{1-r} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{9}{10} \times \frac{10}{9} = 1$

Proof 04

If they are not equal, what is the number between them (in a continuous scale, there has to, if they are not equal)?

Concepts

Indeterminate vs undefined

When something is **undefined**, this means that there are no solutions.

$$\frac{1}{0} = ?$$

$$0 \times ? = 1$$

Which number is to be multiplied with 0 to get 1? There is no such number. That's why $\frac{1}{0} = ?$ is undefined.

However, when something in **indeterminate**, this means that there are infinitely many solutions to the question.

$$\frac{0}{0} = ?$$

$$0 \times ? = 0$$

Which number is to be multiplied with 0 to get 0. You can multiply any number, including 0. There are infinite solutions. So it is indeterminate.

Factorials

Why 0!=1?

Mathematical Proof

$n! = n(n-1)!$

$\Rightarrow 1! = 1(1-1)!$

$\Rightarrow 1 = 1 \times 0!$

$\Rightarrow 1 = 0!$

Logical Proof

1 is multiplicative identity, so before any multiplication starts, we have 1. Also in multiplication, nothing means 1, not 0.

e^x

Euler Formula

$$e^{ix} = \cos x + i \sin x,$$

Proof

$$f(\theta) = \frac{cos\theta + isin\theta}{e^{\theta}} = e^{-i\theta}(cos\theta + isin\theta)$$

Now, differentiating,

$$f'(\theta) = e^{-i\theta}(icos\theta - sin\theta) - ie^{-i\theta}(icos\theta + sin\theta) = 0$$

Thus, $f(\theta)$ is a constant.

Now, $f(0) = 1$, so $f(\theta) = 1$ for all real θ, and thus

$$e^{i\theta} = \cos\theta + i \sin\theta,$$

Euler Identity

e^x derivative

Source: Quora

Using series

$$y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$y' = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Using fundamental principle

$$
\begin{aligned}
y &= e^x \\
y' = \frac{d}{dx}y &= \frac{d}{dx}e^x \\
&= lim_{h\to 0}\frac{e^{x+h} - e^x}{h} \\
&= lim_{h\to 0}\frac{e^x(e^h - 1}{h} \qquad (\#eq:exdiff) \qquad (1) \\
&= lim_{h\to 0}e^x \cdot \frac{e^h - 1}{h} \\
&= lim_{h\to 0} \cdot lim_{h\to 0}\frac{e^h - 1}{h} \\
&= e^x(1) = e^x
\end{aligned}
$$

Intuitive Derivation

Source

The function which is its own derivative is the fundamental idea here, and the logical starting point. This function is the exponential function $exp(z)$ which we can construct based solely on that defining property, and from this function we can derive the constant $e = exp(1)$ as well as its period $2\pi i$, which is the reason π s so important), and prove various expressions for it, including

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = lim_{h\to\infty}\left(1 + \frac{1}{n}\right)^n$$

So the "compound interest" aspect is the minor side effect, not the heart of the logical structure. It is often the case that history takes a meandering approach to discovery, and not the most efficient or logical route. The fact that Bernoulli stumbled upon e initially as the limit of $(1 + 1/n)^n$ is incidental.

Let me elaborate a little bit.

You should begin by seeking a function which satisfies $f' = f$

This is a very natural and very powerful idea, since plenty of differential equations which arise in math, physics and engineering can be solved using such a function (harmonic oscillators, the normal distribution and many more).

Once you hit upon the idea of finding such a function, you can build it as follows. You start with the simplest function

$$f(x) = 1$$

$f(x) = 0$ is also its own derivative, and this is not useful or enlightening).

Unfortunately the derivative is 0, not f itself. What has a derivative of 1? Well, x does, so let's add it in:

$$f(x) = 1 + x$$

e Alternate Expansion

$$e = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \left(\frac{1}{n}\right) + \frac{n \cdot (n-1)}{2!} \cdot \left(\frac{1}{n}\right)^2 + \frac{n \cdot (n-1) \cdot (n-2)}{3!} \cdot \left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n$$

Then

$$e = 1 + 1 + \frac{(1 - \frac{1}{n})}{2!} + \frac{(1 - \frac{1}{n}) \cdot (1 - \frac{2}{n})}{3!} + \cdots + \frac{1}{n^n}$$

Using limit,

$$\lim_{x \to 2} \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Not only is it much easier to compute the terms of this infinite series and add as many of them as we please, but the sum will approach its limiting value much faster than when computing $(1 + \frac{1}{n})^n$ directly

...

Alternative to Pascal's Triangle

$$^nC_k = \frac{n!}{k!(n-k)!}$$

one way to understand which is in how many ways can we get k items out of n items.

A coin is tossed 4 items. We can 3 heads in $^4C_3$ ways.

HHTH or HTHH etc.